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Vertex- Identification on Graphs with Twins

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Abstract: Recently, several vertex identifying notions were introduced (identifying coloring, lid-coloring, ...), these notions were inspired by identifying codes. All of them, as well as original identifying code, are based on separating two vertices according to some conditions on their closed neighborhood. Therefore, twins can not be identified. So most of known results focus on twin-free graph. Here, we show how twins can modify optimal value of vertex-identifying parameters for identifying coloring and locally identifying coloring.

Key Words: Identifying coloring, locally identifying coloring, twins, separating.

MSC(2010): 05C15, 68R10, 94C15

Résumé : Récemment, plusieurs notions d'identification des sommets ont été introduites (coloration identifiante, lid-coloration, ...), ces notions ont été inspirées des codes identifiants. Toutes ces colorations, ainsi que le code identifiant original, sont basés sur la séparation de deux sommets selon certaines conditions sur leur voisinage fermé. Par conséquent, les jumeaux ne peuvent pas être identifiés. Ainsi, la plupart des résultats connus se concentrent sur un graphe sans jumeaux. Dans cet article, nous montrons comment les jumeaux peuvent modifier la valeur optimale des paramètres d'identification des sommets pour la coloration identifiante et la coloration localement identifiante

Mots clés : Coloration identifiante, coloration localement identifiante, jumeaux, séparation.

1 Introduction

In this paper, we are interested in vertex-colorings, distinguishing the vertices of a graph. Given a graph G and a coloring c, a pair of vertices u and v of G is *identified* if and only if $c(N[u]) \neq c(N[v])$ (where $N[u]$ denotes the closed neighborhood of u). Two vertices u and v with $N[u] = N[v]$ are called twins. Observe that two twins cannot be identified. The notion of identifying coloring was introduced Eric Duchêne and Jullien Moncel, during the discrete week of Institute Fourier at Grenoble in 2009.

Clearly, an identifying coloring of G exists if and only if G has no twins.

Since then, several authors [1, 2] introduced two notions of *locally identifying coloring* where only pairs of adjacent vertices must be identified. Moreover, in order to incorporate graphs with twins, the identifying condition concerns only pairs of non-twin vertices.

Since only few results are known on identifying coloring (see [6]), and in order to be uniform, we propose here to modify the definition of identifying coloring of a graph as a vertex coloring such that any pair of **non-twin** vertices are identified.

In [1], a *relaxed locally identifying coloring*, *rlid-coloring* for short, of a graph $G = (V, E)$ is defined as a mapping $c : V \to \mathbb{N}$ such that any pair of **adjacent** non-twin vertices is identified. Parreau et al. [2] introduced the notion of *locally identifying coloring*, *lid-coloring* for short, as a rlid-coloring c which is proper; that is, $c(u) \neq c(v)$ for all pair of adjacent vertices u, v.

Given a graph G, $\chi_{id}(G)$ (respectively $\chi_{lid}(G), \chi_{rlid}(G)$) denotes the smallest number of colors needed to have an identifying coloring (resp. lid-coloring, rlid-coloring) of G.

The χ_{lid} is the most studied of these parameters, see for instance [2, 3, 4]. Nevertheless, except in [1], most of the results concern twin-free graphs. The aim of this paper is to show that twins may have significant influence on these parameters.

In order to state our results, we will need additional definitions. Let $\mathcal{R}$ be the equivalence relation defined as follows: for all vertices $u, v \in V(G)$, we have $u\mathcal{R}v$ if only if $N[u] = N[v]$. Denote by $G\backslash\mathcal{R}$, the maximal twin-free subgraph of G (that is the quotient of G by relation $\mathcal{R}$). The number of equivalence classes having at least two vertices in G is denoted by $t(G)$. Let $T(G)$ be the cardinality of a largest equivalence-class.

In [1], the authors proved the following theorem :

Theorem 1 *Let G be a graph. Then we have*

$$\chi_{rlid}(G\backslash\mathcal{R}) - t(G) \leq \chi_{rlid}(G) \leq \chi_{rlid}(G\backslash\mathcal{R}).$$

Moreover, the authors in [1] exhibit graphs for which the bounds are tight. In this paper, we give analogous results for identifying colorings and lid-colorings.

Theorem 2 *Let G be a graph. Then we have*

$$\chi_{id}(G\backslash\mathcal{R}) - t(G) \leq \chi_{id}(G) \leq \chi_{id}(G\backslash\mathcal{R}).$$

Theorem 3 *Let G be a graph. Then we have*

$$\chi_{lid}(G\backslash\mathcal{R}) - t(G) \leq \chi_{lid}(G) \leq \chi_{lid}(G\backslash\mathcal{R}) + (T(G) - 1)t(G).$$

Proofs of Theorems 2 and 3 are given in Section 2. In Section 3, we exhibit graphs for which the bounds in Theorems 2 and 3 are tight.

2 Proofs of bounds

Preuve. (of Theorem 2.)
We present here a proof similar to the proof of Theorem 1 given in [1].

Consider an identifying coloring c of $G \setminus \mathcal{R}$. Let us define a coloring c' of G as follows : for each vertex x in $G\backslash\mathcal{R}$ and its twin (if there exists) y, set $c'(x) = c'(y) = c(x)$. Since in G, we are not interested to distinguish the twins then c defines an identifying coloring of G.

Now, let c be an identifying coloring of G using colors $\{1, \ldots, \chi_{id}(G)\}$. Consider the coloring c' defined as follows : $c'(u) = c(u)$ if the vertex u has no twin in G and color the other $t(G)$ vertices of $G\backslash\mathcal{R}$ with different colors $\chi_{id}(G) + 1$ until $\chi_{id}(G) + t(G)$. This coloring gives an identifying coloring of $G\backslash\mathcal{R}$. ■

Preuve. (of Theorem 3)
Now, consider a *lid*-coloring c of $G\backslash\mathcal{R}$ using colors $\{1, \ldots, \chi_{lid}(G\backslash\mathcal{R})\}$. Let c' be a coloring obtained from c as follows: $c'(u) = c(u)$ for all u in $G\backslash\mathcal{R}$. By definition, there are at most $(T(G) - 1).t(G)$ vertices in G which are not in $G\backslash\mathcal{R}$. For each of them assign a distinct color from $\{\chi_{lid}(G\backslash\mathcal{R})+1, \ldots, \chi_{lid}(G\backslash\mathcal{R})+(T(G)-1)t(G)\}$. This coloring gives an lid-coloring of G.

Now, similarly as proof of Theorem 2, let c be a *lid*-coloring of G using colors $\{1, \ldots, \chi_{lid}(G)\}$. Consider the coloring c' defined as follows : $c'(u) = c(u)$ if the vertex u has no twin in G and color the other $t(G)$ vertices of $G\backslash\mathcal{R}$ with different colors $\chi_{lid}(G) + 1$ until $\chi_{lid}(G) + t(G)$. This coloring gives a lid-coloring of $G\backslash\mathcal{R}$. ■

3 Extremal graphs

First define the split graph $H_p = (S_p \cup K_p, E)$ for a given integer p where $S_p = \{s_1, \ldots, s_p\}$ (respectively $K_p = \{k_0, \ldots, k_p\}$) induces a stable (resp. clique). The other edges of H_p are $s_i k_i$ for all $i = 1, \ldots, p$.

Proposition 4 *Let $p \geq 1$ be an integer. We have that*

$$\chi_{id}(H_p) = p + 2 \text{ and } \chi_{lid}(H_p) = 2p + 1.$$

Preuve. The coloring c defined by $c(s_i) = i, c(k_i) = p + 1$ for all $i = 1, \ldots p$ and $c(k_0) = p + 2$, is an identifying coloring of H_p.
Let us prove now that $\chi_{id}(H_p) \geq p+2$. Let c be an identifying coloring of H_p. First observe that $c(s_i) \neq c(s_j)$ for all $i \neq j$. Indeed, otherwise $c(N[k_i]) = c(N[k_j])$ which leads a contradiction. Second, suppose that $c(s_i) = c(k_j)$ for some integers i, j (i can be equal to j). Then $c(N[k_0]) = c(N[k_i])$, a contradiction. To conclude, check that if $|c(K)| = 1$ then $c(N[s_i]) = c(N[k_i])$ for all i, a contradiction which completes the proof of $\chi_{id}(H_p) \geq p + 2$.

Any coloring using $2p + 1$ distinct colors is a lid-coloring of H_p.
Let us prove now that $\chi_{lid}(H_p) \geq 2p + 1$. Let c be an lid-coloring of H_p. As previously, we have $c(s_i) \neq c(s_j)$ for all $i \neq j$ else $c(N[k_i]) = c(N[k_j])$ and k_i and k_j are adjacent. Again, $c(s_i) \neq c(k_j)$ for all pair i, j, otherwise $c(N[k_0]) = c(N[k_i])$ for some $i \neq 0$, a contradiction. To conclude, since K is a clique, then $|c(K)| = p + 1$. ■

Consider the first extension $H_{2^a}^{ext} = (S_{a.2^a} \cup K_{2^a}, E)$ for some integer $a \geq 1$ where K_{2^a} induces a clique. One may define the vertices of $K_{2^a} = \{k_{\mathcal{E}} | \mathcal{E} \subseteq \{1, \ldots, a\}\}$. Now define $S_{a.2^{a-1}} = \{s_{\mathcal{E}_i} | \mathcal{E} \subseteq \{1, \ldots, a\}$ and $i \in \mathcal{E}\}$. Observe that $|K_{2^a}| = 2^a$ and $|S_{a.2^{a-1}}| = a.2^{a-1}$. The other edges of $H_{2^a}^{ext}$ are $s_{\mathcal{E}_i} k_{\mathcal{E}}$ for all $i \in \mathcal{E}$ and $s_{\mathcal{E}_i} s_{\mathcal{E}_j}$ for all $i, j \in \mathcal{E}$.
We remark that $H_{2^a}^{ext} \backslash \mathcal{R} = H_{2^a-1}$ with $t(H_{2^a}^{ext}) = 2^a - 1 - a$ and $T(H_{2^a}^{ext}) = a$.

Proposition 5 *Let $a \geq 1$ be an integer. We have that*

$$\chi_{id}(H_{2^a}^{ext}) = a + 2 \text{ and } \chi_{lid}(H_{2^a}^{ext}) = a + 2^a.$$

Preuve. The coloring c defined by $c(s_{\mathcal{E}_i}) = i, c(k_{\mathcal{E}}) = a + 1$ for all $\mathcal{E} \subseteq \{1, \ldots, a\}$ and for all $i \in \mathcal{E}$ and $c(k_0) = a + 2$, is an identifying coloring of $H_{2^a}^{ext}$.
By Theorem 2, we have $\chi_{id}(H_{2^a}^{ext}) \geq \chi_{id}(H_{2^a}^{ext} \backslash \mathcal{R}) - t(H_{2^a}^{ext}) = 2^a - 1 + 2 - (2^a - 1 - a) = a + 2$.

For the lid-coloring the proof is similar except that we need distinct colors for each vertex in the clique K_{2^a}. ■

The two previous propositions show that lower bound of Theorems 2 and 3 are tight for graph $H_{2^a}^{ext}$. The upper bound of Theorem 2 is reached for any twin-free graph.

Given integers $p \geq t \geq 1$ and $T \geq 1$, consider the graph $H_p^{(T,t)}$ obtained from H_p by adding $T - 1$ twins to all vertices k_i for all $i = 1, ..., t$.
Remark that $H_p^{(T,t)} \backslash \mathcal{R} = H_p$ with $t(H_p^{(T,t)}) = t$ and $T(H_p^{(T,t)}) = T$.

Proposition 6 *Let $a \geq 1$ be an integer. We have that*

$$\chi_{lid}(H_p^{(T,t)}) = 2p + 1 + (T - 1).t.$$

Preuve. Any coloring using $2p + 1 + (T - 1).t$ distinct colors is a lid-coloring of $H_p^{(T,t)}$.
The proof of $\chi_{lid}(H_p^{(T,t)}) \geq 2p + 1 + (T - 1).t$ follows the one of Proposition 4. ■

For all triples of integers $p \geq t \geq 1$ and $T \geq 1$, Property 6 shows that the upper bound of Theorem 3 is reached.

4 Concluding Remarks

The main motivation of the present paper is to point out that twins may play a crucial role in identifying coloring problems using closed neighborhoods. Probably it would be too difficult to re-consider all known results on twin-free graphs. But there are some special classes of graphs (e.g. split graphs) for which this work remains attractive.
Instead of coloring, one may ask what happens for identifying codes. An identifying code [5] is a subset of vertices C, such that for any pair of distinct vertices u, v, $N[u] \cap C \neq N[v] \cap C$. This is the classical definition, and clearly, a graph having twins does not admit an identifying code.

Consider, now the new definition where the condition $N[u] \cap C \neq N[v] \cap C$ has to be verified only for non-twin pair of vertices.
It is not too difficult to see that the size of a minimum identifying codes in a graph G with new definition is equal to the size of a minimum identifying codes in $G \backslash \mathcal{R}$. Therefore, it is not restrictive for identifying codes to consider only twin-free graphs.
Now, for coloring versions of identifying problems one may consider a *weighted* version. Given a graph G and a weight function $w : V \to \mathbb{N}$, a mapping $c : V \to \mathcal{P}(\mathbb{N})$ is an *weighted-identifying coloring* of G if and only if $|c(u)| \leq w(u)$ for all vertices u and all distinct pairs of non-twin vertices are identified.
Observe that an optimal value for a pair (w, G) is the same than the optimal value for the pair $(w', G\backslash\mathcal{R})$ where $w'(u) = w(u) + T(u) - 1$ where $T(u)$ is the number of twins of u. So for this new definition one may focus only on twin-free graphs.

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